Lecture 1: Root systems. 3/26/19

§1. Roots.

We must start on the level of Lie algebras, rather than Lie groups.

Definitions

- A complex, finite dimensional Lie algebra $\mathfrak{g}$ is semisimple if $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]:=\{[X,Y] \mid X,Y\in\mathfrak{g}\}$.

Example $\mathfrak{sl}_n(\mathbb{C})=\{X\in M_{n\times n}(\mathbb{C}) \mid \operatorname{Tr} X=0\}$ is semisimple. ($[X,Y]=XY-YX$.)

Def A Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$ is a subalg $\mathfrak{h}$ such that

- $\mathfrak{h}$ is nilpotent, which means $[\mathfrak{h},[\mathfrak{h},\dots,[\mathfrak{h},\mathfrak{h}]\dots]]$ (n brackets) is zero for some n.
- $\mathfrak{h}=N_{\mathfrak{g}}(\mathfrak{h})$ $(:=\{X\in\mathfrak{g} \mid [X,\mathfrak{h}]\subseteq\mathfrak{h}\})$ (i.e., $\mathfrak{h}$ is self-normalizing)

It's usually hard to use this definition, so here are some facts. I'll give some examples of everything in a moment.

Facts

- Cartan subalgebras are abelian (i.e., $[\mathfrak{h},\mathfrak{h}]=0$)
- All Cartans are conjugate by an automorphism of $\mathfrak{g}$, hence they all have the same dimension.

Def $\operatorname{rk}(\mathfrak{g}):=\dim(\text{any Cartan})$, the rank of $\mathfrak{g}$.

Notation Write $\operatorname{ad}(X)(Y)=[X,Y]$.

Then $\operatorname{ad}(X):\mathfrak{g}\to\mathfrak{g}$ is a linear map and so is $\operatorname{ad}:\mathfrak{g}\to\operatorname{End}_{\mathbb{C}}(\mathfrak{g})$.

In fact, $\operatorname{ad}([X,Y])=\operatorname{ad}(X)\circ\operatorname{ad}(Y)-\operatorname{ad}(Y)\circ\operatorname{ad}(X)$, so ad is a representation $\mathfrak{g}\to\operatorname{End}_{\mathbb{C}}(\mathfrak{g})$.

Fact $\operatorname{ad}(\mathfrak{h})$ is simultaneously diagonalizable, meaning there's a basis $\{X_i\}_{i=1}^{\dim\mathfrak{g}}$ of $\mathfrak{g}$ such that:

$$\forall H\in\mathfrak{h},\ \forall i,\ \exists \lambda_{H,i}\in\mathbb{C} \text{ s.t. } \operatorname{ad}(H)(X_i)=[H,X_i]=\lambda_{H,i}X_i$$

Note that the map $\alpha_i: H\mapsto\lambda_{H,i}$ is a linear map $\mathfrak{h}\to\mathbb{C}$, i.e., a character of $\mathfrak{h}$ (because $[aH_1+bH_2,X_i]=a[H_1,X_i]+b[H_2,X_i]$ $=(a\lambda_{H_1,i}+b\lambda_{H_2,i})X_i$.)

Definition The nonzero α_i's are called the roots of $\mathfrak{g}$.

Example ($\mathfrak{sl}_3$)

Cartan $\mathfrak{h}=\begin{pmatrix}*&&\\&*&\\&&*\end{pmatrix}$ (This is the Lie algebra of the standard maximal torus in $SL_3(\mathbb{C})$.) Note $\dim\mathfrak{h}=2$.

The basis of X's is: The six elementary matrices E_{ij} for $i\neq j$. $E_{12}=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$, $E_{13}=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$, ..., along with any basis for $\mathfrak{h}$.

We check:

$$\operatorname{ad}\left(\begin{pmatrix}a_1&&\\&a_2&\\&&a_3\end{pmatrix}\right)(E_{12})=\left[\begin{pmatrix}a_1&&\\&a_2&\\&&a_3\end{pmatrix},E_{12}\right]=\begin{pmatrix}a_1&&\\&a_2&\\&&a_3\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}-\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}a_1&&\\&a_2&\\&&a_3\end{pmatrix}=\begin{pmatrix}0&a_1&0\\0&0&0\\0&0&0\end{pmatrix}-\begin{pmatrix}0&a_2&0\\0&0&0\\0&0&0\end{pmatrix}=(a_1-a_2)E_{12}.$$

So if we define $\alpha_{ij}:=$ the root for $\mathbb{C}\cdot E_{ij}$, then $\alpha_{12}\begin{pmatrix}a_1&&\\&a_2&\\&&a_3\end{pmatrix}=a_1-a_2$.

Similarly, $\alpha_{ij}\begin{pmatrix}a_1&&\\&a_2&\\&&a_3\end{pmatrix}=a_i-a_j$, for $i\neq j$, are the other roots.

Example ($\mathfrak{sl}_n$).

$\mathfrak{h}=\begin{pmatrix}*&&\\&\ddots&\\&&*\end{pmatrix}$. The simultaneous eigenbasis consists of E_{ij}, $i,j=1,\dots,n$, $i\neq j$, along with any basis for $\mathfrak{h}$.

The roots are $\alpha_{ij}=$ simultaneous eigenvalue of $\mathfrak{h}$ on E_{ij}, given by $\alpha_{ij}\begin{pmatrix}a_1&&\\&\ddots&\\&&a_n\end{pmatrix}=a_i-a_j$.

Fact A subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ is a Cartan $\Leftrightarrow$ it is maximal abelian and $\operatorname{ad}(\mathfrak{h})$ is simultaneously diagonalizable.

("Maximal abelian" means it is contained in no larger abelian subalgebra, not that it has maximal dimension among abelian subalgebras.)

§2. Properties of roots.

Fix a Cartan $\mathfrak{h}$ in a semisimple Lie algebra $\mathfrak{g}$.

Let Φ be the (finite) set of roots of $\mathfrak{h}$. Note $\Phi \subseteq \mathfrak{h}^\vee := \mathrm{Hom}_\mathbb{C}(\mathfrak{h}, \mathbb{C})$.

(1) For $\alpha \in \Phi$, let $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \ \forall\, H \in \mathfrak{h}\}$ be the α-eigenspace. Then $\dim \mathfrak{g}_\alpha = 1$.

(In the $\mathfrak{sl}_n$ example, $\mathfrak{g}_{\alpha_{ij}} = \mathbb{C} \cdot E_{ij}$.)

(2) (Root space decomposition) $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$.

(3) If α is a root, then so is $-\alpha$.

(In the $\mathfrak{sl}_n$ example, $-\alpha_{ij} = \alpha_{ji}$.)

(4) Φ spans $\mathfrak{h}^\vee$.

(In the $\mathfrak{sl}_n$ example, $\{\alpha_{i,i+1} \mid i = 1, \dots, n-1\}$ is a spanning set.)

Something more general than (3) is true. To explain it, we need to talk about the Killing form, which one needs anyway to justify the facts which have been said already.

<u>Def</u> The <u>Killing form</u> is the symmetric bilinear form on $\mathfrak{g}$

$$\langle X, Y \rangle = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)).$$

<u>Note</u> $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ is a linear endomorphism of the vector space $\mathfrak{g}$, so we can just take the trace as usual. The Killing form is symmetric because $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

<u>Thm</u> (Cartan) $\mathfrak{g}$ is semisimple $\Leftrightarrow$ $\langle \cdot, \cdot \rangle$ is nondegenerate.

<u>Fact</u> If $\mathfrak{g}$ is semisimple, then $\langle \cdot, \cdot \rangle$ restricts to a perfect pairing on any Cartan $\mathfrak{h}$.

Therefore we get a canonical isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^\vee$ by $H \mapsto \langle H, \cdot \rangle$, and hence also a perfect pairing $\langle \cdot, \cdot \rangle : \mathfrak{h}^\vee \times \mathfrak{h}^\vee \to \mathbb{C}$, given explicitly by

$$\langle \langle H_1, \cdot \rangle, \langle H_2, \cdot \rangle \rangle = \langle H_1, H_2 \rangle.$$

(5) If α, β are roots, so is the reflection of β across the hyperplane perpendicular to α, i.e.,

$$\alpha, \beta \in \Phi \Rightarrow s_\alpha(\beta) := \beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha \in \Phi.$$

<u>Example</u> Let's compute $s_{\alpha_{12}}(\alpha_{23})$ for $\mathfrak{sl}_3$.

$$s_{\alpha_{12}}(\alpha_{23}) = \alpha_{23} - 2 \frac{\langle \alpha_{12}, \alpha_{23} \rangle^{?}}{\langle \alpha_{12}, \alpha_{12} \rangle_{?}} \alpha_{12}.$$

To compute the Killing form, we need to make explicit the identification $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^\vee$ by $H \mapsto \langle H, \cdot \rangle$, and so we need to compute $\langle \cdot, \cdot \rangle$ on a basis of $\mathfrak{h}$.

Let $H_{12} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$, $H_{23} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$. A basis of $\mathfrak{sl}_3$ consists of H_{12}, H_{23}, E_{ij} for $i \neq j$). One computes easily:

$$\mathrm{ad}(H_{12})(E_{ij}) = \begin{cases} 2E_{12} & i,j = 1,2 \\ 1E_{13} & i,j = 1,3 \\ -1E_{23} & i,j = 2,3 \\ -2E_{21} & i,j = 2,1 \\ -1E_{31} & i,j = 3,1 \\ 1E_{32} & i,j = 3,2, \end{cases} \qquad \mathrm{ad}(H_{23})(E_{ij}) = \begin{cases} -1E_{12} & i,j = 1,2 \\ 1E_{13} & i,j = 1,3 \\ 2E_{23} & i,j = 2,3 \\ 1E_{21} & i,j = 2,1 \\ -1E_{31} & i,j = 3,1 \\ -2E_{32} & i,j = 3,2, \end{cases} \qquad \mathrm{ad}(H_{12})(\mathfrak{h}) = \mathrm{ad}(H_{23})(\mathfrak{h}) = 0.$$

Therefore,

$$\langle H_{12}, H_{23} \rangle = \mathrm{Tr}(\mathrm{ad}(H_{12}) \circ \mathrm{ad}(H_{23})) = (2)(-1) + (1)(1) + (-1)(2) + (-2)(1) + (-1)(-1) + (1)(2) = -2 + 1 - 2 - 2 + 1 - 2 = -6.$$

Just multiply the numbers we got above.

Similarly,

$$\langle H_{12}, H_{12} \rangle = 4 + 1 + 1 + 4 + 1 + 1 = 12 = \langle H_{23}, H_{23} \rangle.$$

Let $\varphi_{12} = \langle H_{12}, \cdot \rangle \in \mathfrak{h}^\vee$, $\varphi_{23} = \langle H_{23}, \cdot \rangle \in \mathfrak{h}^\vee$. Then

$$\varphi_{12}\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = \varphi\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & -a_1-a_2 \end{pmatrix}$$

$$= \varphi_{12}\begin{pmatrix} a_1 & & \\ & -a_1 & \\ & & 0 \end{pmatrix} + \varphi_{12}\begin{pmatrix} 0 & & \\ & a_1+a_2 & \\ & & -a_1-a_2 \end{pmatrix}$$

$$= a_1 \varphi_{12}(H_{12}) + (a_1+a_2)\varphi_{12}(H_{23})$$

$$= a_1 \langle H_{12}, H_{12} \rangle + (a_1+a_2)\langle H_{12}, H_{23} \rangle$$

$$= 12 a_1 - 6(a_1+a_2)$$

$$= 6(a_1 - a_2)$$

$$= 6\alpha_{12}\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}$$

So, $\varphi_{12} = 6\alpha_{12}$.
Similarly, $\varphi_{23} = 6\alpha_{23}$.
Therefore,

$$\langle \alpha_{12}, \alpha_{23} \rangle = \langle \tfrac{1}{6}\varphi_{12}, \tfrac{1}{6}\varphi_{23} \rangle = \tfrac{1}{36}\langle H_{12}, H_{23} \rangle = \tfrac{1}{36}(-6) = \tfrac{-1}{6},$$
$$\langle \alpha_{12}, \alpha_{12} \rangle = \langle \tfrac{1}{6}\varphi_{12}, \tfrac{1}{6}\varphi_{12} \rangle = \tfrac{1}{36}\langle \varphi_{12}, \varphi_{12} \rangle = \tfrac{1}{36}(12) = \tfrac{1}{3},$$

and so

$$\frac{2\langle \alpha_{12}, \alpha_{23} \rangle}{\langle \alpha_{12}, \alpha_{12} \rangle} = \frac{2\cdot(-1/6)}{1/3} = -1$$

Thus we have computed the number we were looking for, and we can substitute back into the definition of $s_{\alpha_{12}}(\alpha_{23})$:

$$s_{\alpha_{12}}(\alpha_{23}) = \alpha_{23} - \frac{2\langle \alpha_{12}, \alpha_{23} \rangle}{\langle \alpha_{12}, \alpha_{12} \rangle}\alpha_{12} = \alpha_{23} + \alpha_{12}$$

But this should be a root, i.e., $s_{\alpha_{12}}(\alpha_{23})$ should equal α_{ij} for some i,j. However, we see that

$$(\alpha_{23} + \alpha_{12})\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_2 - a_3 + a_1 - a_2 = a_1 - a_3 = \alpha_{13}\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}.$$

So

$$\boxed{s_{\alpha_{12}}(\alpha_{23}) = \alpha_{13}.}$$

We can compute $s_{\alpha_{12}}$ on any other root as well:

$$s_{\alpha_{12}}(\alpha_{ij}) = \begin{cases} -\alpha_{12} & i,j = 1,2 \\ \alpha_{23} & i,j = 1,3 \\ \alpha_{13} & i,j = 2,3 \leftarrow s_{\alpha_{12}}^2 = \mathrm{id} \text{ b/c it's a reflection} \\ \alpha_{12} & i,j = 2,1 \\ -\alpha_{23} & i,j = 3,1 \leftarrow \text{b/c } s_\alpha \text{ is linear, so } s_\alpha(-\beta) = -s_\alpha(\beta). \\ -\alpha_{13} & i,j = 3,2 \end{cases}$$

In general, $s_\alpha \in O(\mathfrak{h}^\vee, \langle\cdot,\cdot\rangle)$ (the orthogonal group), $s_\alpha(\varphi) := \varphi - \frac{2\langle \alpha, \varphi \rangle}{\langle \alpha, \alpha \rangle}\alpha$, for general $\varphi \in \mathfrak{h}^\vee$.

<u>Def</u> $W_\Phi :=$ subgroup of $O(\mathfrak{h}^\vee, \langle\cdot,\cdot\rangle)$ generated by s_α's for $\alpha \in \Phi$. W_Φ is called the <u>Weyl group</u> of Φ.

(6) W_Φ is a finite group.

<u>Note</u> $\frac{2\langle \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 2$, so $s_\alpha(\alpha) = \alpha - 2\alpha = -\alpha$. Therefore $(5) \Rightarrow (3)$.

(7) $\forall \alpha,\beta\in\Phi$, $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}$, and in fact,

(7)' $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\in\{0,\pm1,\pm2,\pm3\}$. In particular,

(8) $\alpha\in\Phi\Rightarrow n\alpha\notin\Phi$ for any $n\neq\pm1$ (because if $|n|>1$, then $\frac{2\langle\alpha,n\alpha\rangle}{\langle\alpha,\alpha\rangle}=2|n|\geq 4$.)

(9) The $\mathbb{Z}$-span of Φ is a lattice in the $\mathbb{R}$-span of Φ. $\langle\cdot,\cdot\rangle$ is real-valued and positive definite on the $\mathbb{R}$-span of Φ.

Def A <u>root system</u> is an $\mathbb{R}$-vector space V together with a pos. def. symmetric pairing $\langle\cdot,\cdot\rangle$ and a subset $\Phi\subseteq V$ s.t.

(a) $0\notin\Phi$

(In practice, this will be the real form of our $\mathfrak{h}^\vee$ given by the $\mathbb{R}$-span of Φ)

(b) Φ spans V

(c) $\forall\alpha,\beta\in\Phi$, $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}$

(d) $\forall\alpha,\beta\in\Phi$, $s_\alpha(\beta)=\beta-\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}\alpha\in\Phi$

Φ is <u>reduced</u> if also

(e) $\alpha\in\Phi\Rightarrow 2\alpha\notin\Phi$.

<u>Remark</u> With some effort, finiteness of W_Φ (defined the same way) and the fact that $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\in\{0,\pm1,\pm2,\pm3,\pm4\}$, can be shown for general root systems. Furthermore, $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}=\pm4\Leftrightarrow\beta=\pm2\alpha$, so if Φ is reduced, then (8) follows for Φ. (9) follows from the theory of simple roots (see the next section)

<u>Summary</u>

Given a semisimple Lie algebra $\mathfrak{g}$, a Cartan $\mathfrak{h}\subseteq\mathfrak{g}$ is a subalgebra which is maximal abelian and such that $\mathrm{ad}(\mathfrak{h})$ is simultaneously diagonalizable.

We get a reduced root system $(V,\Phi,\langle\cdot,\cdot\rangle)$ by:

- $\Phi=\{\text{nonzero simultaneous eigenvalues of } \mathrm{ad}(\mathfrak{h}) \text{ on } \mathfrak{g}\}\subseteq\mathfrak{h}^\vee$
- $V=\mathbb{R}$-span of Φ (so $V\otimes_{\mathbb{R}}\mathbb{C}=\mathfrak{h}^\vee$).
- $\langle\cdot,\cdot\rangle$ is induced from the Killing form: $\langle X,Y\rangle=\mathrm{Tr}(\mathrm{ad}(X)\circ\mathrm{ad}(Y))$ for $X,Y\in\mathfrak{h}$ (the operator $\mathrm{ad}(X)\circ\mathrm{ad}(Y)$ acting on $\mathfrak{g}$) and then $\langle\langle X,\cdot\rangle,\langle Y,\cdot\rangle\rangle:=\langle X,Y\rangle$ puts $\langle\cdot,\cdot\rangle$ on $V\subseteq\mathfrak{h}^\vee$.

Then $\forall\alpha\in\Phi$, $\mathfrak{g}_\alpha=\alpha$-eigenspace (or root space) is 1-dimensional.

§3. Positive and simple roots.

Fix a root system $(V,\Phi,\langle\cdot,\cdot\rangle)$.

Fix any total order "$>$" on V such that:

- $\varphi_1,\varphi_2>0\Rightarrow\varphi_1+\varphi_2>0$
- Exactly one of the following holds for any $\varphi\in V$:

$$\varphi>0,\quad -\varphi>0,\quad \varphi=0.$$

Say φ is <u>positive</u> if $\varphi>0$.

For instance, fixing a basis $\varphi_1,\dots,\varphi_r$ of V, we can take $\sum_{i=1}^r a_i\varphi_i>\sum_{i=1}^r b_i\varphi_i$ if $a_1=b_1,\dots,a_{i-1}=b_{i-1}, a_i>b_i$, some i.

Let $\Phi^+=\{\alpha\in\Phi\mid\alpha>0\}$ – the <u>positive roots</u>. Then $\Phi^-:=\{-\alpha\mid\alpha\in\Phi^+\}$ has $\Phi^-\cup\Phi^+=\Phi$ and $\Phi^-\cap\Phi^+=\emptyset$.

Def A root is <u>simple</u> if it is positive, and cannot be written as a sum of two positive roots.

Let $\Delta\subset\Phi^+$ be the set of simple roots.

Facts
- $\#\Delta = \dim V$.
- Any $\beta \in \Phi^+$ can be (uniquely) written $\beta = \sum_{\alpha\in\Delta} n_\alpha \alpha$, $n_\alpha \geq 0$ $\forall \alpha \in \Delta$. Hence any $\gamma \in \Phi$ can be written as $\gamma = \sum_{\alpha\in\Delta} m_\alpha \alpha$, where the m_α's all have the same sign.
- Since Φ, hence Φ^+, spans V, it follows that Δ is $\mathbb{R}$-linearly independent, and that the n_α's above are unique.

The choice of ordering we started with has an effect on Δ. But:

Thm W_Φ acts simply transitively on the set of all possible Δ's. If $>$ and $\succ$ give the same Δ, then they give the same Φ^+. The lexicographical ordering described above with Δ as a basis (any order on Δ) makes Δ simple.

Example ($\mathfrak{sl}_n$). - $\alpha_{12}, \alpha_{23}, \dots, \alpha_{n-1,n}$ are simple for the lexicographical ordering they induce.

- Note there are $n-1 = \mathrm{rk}(\mathfrak{sl}_n)$ of them.
- Then the positive roots are α_{ij}, $j>i$.
- Note $\alpha_{ij} = \alpha_{i,i+1} + \alpha_{i+1,i+2} + \cdots + \alpha_{j-1,j}$ if $j>i$

Lecture 2. Root systems (continued). 4/2/19

Recall A (finite dimensional) semisimple Lie algebra $\mathfrak{g}$ and a Cartan $\mathfrak{h} \subseteq \mathfrak{g}$ gives rise to a set of roots

$$\Phi := \{\text{nonzero simultaneous eigenvalues of } \mathrm{ad}(\mathfrak{h}) \text{ on } \mathfrak{g}\} \subseteq \mathfrak{h}^\vee.$$

For $\alpha \in \Phi$, the α-eigenspace $\mathfrak{g}_\alpha$ is 1 dim'l, and

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

Then $(V := \mathbb{R}\Phi, \Phi, \langle \cdot, \cdot \rangle := \text{Killing form})$ is a reduced root system. In particular,

$$s_\alpha(\beta) := \beta - \underbrace{\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}}_{0, \pm 1, \pm 2, \pm 3} \alpha \in \Phi \quad \forall \alpha, \beta \in \Phi.$$

The Weyl group $W_\Phi := \langle s_\alpha \rangle \subseteq O(V, \langle \cdot, \cdot \rangle)$ is finite.

A total order "$>$" on V, respecting $+$ and $-$, determines $\Phi^+ = \{\alpha \in \Phi \mid \alpha > 0\}$ and $\Delta = \{\text{simple roots}\} \subseteq \Phi^+$.

W_Φ acts simply transitively on $\{\Delta\}$, and Δ determines an order.

Δ is a basis for V.

§4. Dynkin Diagrams

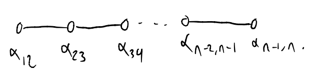
Fact If $\alpha, \beta \in \Delta$, $\alpha \neq \beta$, then $\langle \alpha, \beta \rangle \leq 0$ (hence $\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \{0, -1, -2, -3\}$ in this case.)

Def The Dynkin diagram is the graph with $\{\text{vertices}\} = \Delta$, and $\max\left\{\frac{\langle \alpha, \alpha \rangle}{\langle \beta, \beta \rangle}, \frac{\langle \beta, \beta \rangle}{\langle \alpha, \alpha \rangle}\right\}$ edges between α and β, directed towards α if $\langle \alpha, \alpha \rangle < \langle \beta, \beta \rangle$. (e.g., $\alpha \Rrightarrow \beta$ means α is longer than β).

Example ($\mathfrak{sl}_n$) Recall $\alpha_{ij}\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = a_i - a_j$, $\Delta = \{\alpha_{i,i+1} \mid i = 1, \dots, n-1\}$. One shows

$$\frac{2\langle \alpha_{i,i+1}, \alpha_{j,j+1} \rangle}{\langle \alpha_{i,i+1}, \alpha_{i,i+1} \rangle} = \begin{cases} -1 & i = j+1 \text{ or } j = i+1 \\ 2 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the Dynkin diagram is

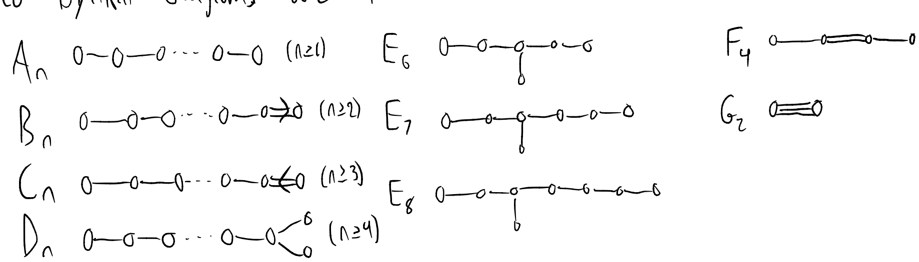


(A_{n-1})

Def Φ is reducible if there are subspaces $V_1, V_2 \subseteq V$ with subsets $\Phi_i \subseteq V_i$ such that $V = V_1 \oplus V_2$, $\Phi = \Phi_1 \sqcup \Phi_2$, and $(V_i, \Phi_i, \langle \cdot, \cdot \rangle)$ are root systems. Otherwise, Φ is irreducible.

Rmk Simple Lie algebras $\longleftrightarrow$ reduced irreducible root systems $\longleftrightarrow$ Connected Dynkin diagrams.

Thm
- Two reduced, irreducible root systems having the same Dynkin diagram are isomorphic (meaning there's a linear isomorphism between vector spaces preserving the form and sending roots bijectively to roots.)
- The possible connected Dynkin diagrams are the usual ones:



- The Dynkin diagram of $\mathfrak{g}$ determines $\mathfrak{g}$ up to isomorphism.

§5. Recovering $\mathfrak{g}$ From its Dynkin Diagram

We want to find the root system given just the simple roots. We'll use the following tool:

Def Let $\alpha, \beta \in \Phi$. The root string of α containing β is
$$\mathrm{Str}_\alpha(\beta) := \{\beta + n\alpha \mid n \in \mathbb{Z} \text{ and } \beta + n\alpha \in \Phi\}.$$

Facts
- $\mathrm{Str}_\alpha(\beta)$ has no gaps.
- Let $p, q \in \mathbb{Z}$ s.t. $\beta + n\alpha \in \mathrm{Str}_\alpha(\beta) \iff p \le n \le q$. Then
$$p + q = -\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}.$$
- If $\alpha, \beta \in \Delta$, then $\alpha - \beta \notin \Phi$, and so $p = 0$ and $q = -\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$.
- If $\gamma = \sum_{\alpha\in\Delta} n_\alpha \alpha \in \Phi^+$, with $\sum n_\alpha > 1$, then we can find $\beta \in \Delta$ such that $\gamma - \beta \in \Phi$.

These facts will let us recover Φ. How about the Lie bracket?

Facts
- If $\alpha, \beta, \alpha+\beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
- $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}\cdot H_\alpha \subseteq \mathfrak{h}^\vee$, where $\langle H_\alpha, \cdot\rangle = \alpha(\cdot)$. (In fact $[E_\alpha, E_{-\alpha}] = \langle E_\alpha, E_{-\alpha}\rangle H_\alpha$.)

Now for $\alpha \in \Delta$, let:
- $h_\alpha = \frac{2}{\langle\alpha,\alpha\rangle} H_\alpha$,
- $e_\alpha \in \mathfrak{g}_\alpha$ be any nonzero vector,
- $f_\alpha \in \mathfrak{g}_{-\alpha}$ s.t. $\langle e_\alpha, f_\alpha\rangle = \frac{2}{\langle\alpha,\alpha\rangle}$.

With these choices, $\mathbb{C}\cdot h_\alpha + \mathbb{C}e_\alpha + \mathbb{C}f_\alpha \cong \mathfrak{sl}_2$; $h_\alpha \mapsto \begin{pmatrix}1&\\&-1\end{pmatrix}$, $e_\alpha \mapsto \begin{pmatrix}0&1\\0&0\end{pmatrix}$, $f_\alpha \mapsto \begin{pmatrix}0&0\\1&0\end{pmatrix}$, and
- $[h_\alpha, e_\alpha] = \frac{2}{\langle\alpha,\alpha\rangle}\alpha(H_\alpha)e_\alpha = \frac{2}{\langle\alpha,\alpha\rangle}\langle H_\alpha, H_\alpha\rangle e_\alpha = \frac{2}{\langle\alpha,\alpha\rangle}\langle\langle H_\alpha,\cdot\rangle, \langle H_\alpha,\cdot\rangle\rangle e_\alpha = \frac{2}{\langle\alpha,\alpha\rangle}\langle\alpha,\alpha\rangle e_\alpha = 2e_\alpha$,
- $[h_\alpha, f_\alpha] = -2f_\alpha$,
- $[e_\alpha, f_\alpha] = \langle e_\alpha, f_\alpha\rangle H_\alpha = \frac{2}{\langle\alpha,\alpha\rangle} H_\alpha = h_\alpha$.

Thm (Serre) The vectors $\{h_\alpha, e_\alpha, f_\alpha \mid \alpha \in \Delta\}$ generate $\mathfrak{g}$ as a Lie algebra with precisely the following relations:

(1) $[h_\alpha, h_\beta] = 0$

(2) $[e_\alpha, f_\beta] = \delta_{\alpha\beta} h_\alpha$

(3) $[h_\alpha, e_\beta] = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} e_\beta$

(4) $[h_\alpha, f_\beta] = -\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} f_\beta$ (1 plus length of $\mathrm{Str}_\alpha(\beta)$)

(5) $(\mathrm{Ad}(e_\alpha))^{1-(2\langle\alpha,\beta\rangle/\langle\alpha,\alpha\rangle)} e_\beta = 0$

(6) $(\mathrm{Ad}(f_\alpha))^{1-(2\langle\alpha,\beta\rangle/\langle\alpha,\alpha\rangle)} f_\beta = 0$

We give an instructive example on the next page.

Example ($\mathfrak{g}_2$) Let $\mathfrak{g}_2$ have Dynkin diagram  (α longer).

Find Φ^+: We have, by the diagram:

$$\frac{2\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} = -1, \qquad \frac{2\langle \alpha,\beta\rangle}{\langle \beta,\beta\rangle} = -3.$$

So

$$Str_\alpha(\beta) \overset{p=0,\ q=1}{=} \{\beta, \alpha+\beta\}, \qquad Str_\beta(\alpha) \overset{p=0,\ q=3}{=} \{\alpha, \alpha+\beta, \alpha+2\beta, \alpha+3\beta\}$$

Note that since $\alpha+\beta \in Str_\alpha(\beta)$, we have $Str_\alpha(\alpha+\beta) = Str_\alpha(\beta)$. Similarly, $Str_\beta(\alpha) = Str_\beta(\alpha+\beta) = Str_\beta(\alpha+2\beta) = Str_\alpha(\alpha+3\beta)$
Then we try taking $Str_{\alpha \text{ or } \beta}$(new roots) to generate more roots, repeat until we can't find new ones, then all roots are exhausted. We compute:

$Str_\alpha(\alpha+2\beta)$: $\alpha+\alpha+2\beta = 2(\alpha+\beta) \notin \Phi$ by reducedness; $-\alpha+\alpha+2\beta = 2\beta \notin \Phi$. No new roots.

$Str_\alpha(\alpha+3\beta)$: $-\alpha+\alpha+3\beta = 3\beta \notin \Phi \Rightarrow p=0$. So $p+q = q = -\frac{2\langle\alpha,\alpha+3\beta\rangle}{\langle\alpha,\alpha\rangle} = -\left(\frac{2\langle\alpha,\alpha\rangle}{\langle\alpha,\alpha\rangle} + 3\,\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\right) = -(2-3) = 1$. So $\alpha+\alpha+3\beta = 2\alpha+3\beta \in \Phi$.

This gives us one more root to try:

$Str_\beta(2\alpha+3\beta)$: $-\beta+2\alpha+3\beta = 2(\alpha+\beta) \notin \Phi$, $\beta+2\alpha+3\beta = 2(\alpha+2\beta) \notin \Phi$. No new roots.

We find

$$\Phi^+ = \{\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\}.$$

So $\dim \mathfrak{g}_2 = \#\Delta + 2\cdot\#\Phi^+ = 14$, and a basis is

$$h_\alpha, h_\beta, e_\alpha, e_\beta, [e_\alpha,e_\beta], [e_\beta,[e_\alpha,e_\beta]], [e_\beta,[e_\beta,[e_\alpha,e_\beta]]], [e_\alpha,[e_\beta,[e_\beta,[e_\alpha,e_\beta]]]], \text{ same brackets with } f_\alpha, f_\beta.$$

Serre relations + Jacobi identity give all possible brackets. For example:

$$\begin{aligned}
[f_\alpha,[e_\alpha,e_\beta]] &= -[e_\beta,[f_\alpha,e_\alpha]] - [e_\alpha,[e_\beta,f_\alpha]] \\
&= -[e_\beta, -h_\alpha] - 0 \\
&= -\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} e_\beta \\
&= e_\beta.
\end{aligned}$$

Lecture 3. Reductive Groups. 4/9/19

Fix a field $k \subseteq \mathbb{C}$.

Def A linear algebraic group over a field k is reductive if its unipotent radical (i.e., the largest closed, connected, normal, unipotent subgroup is trivial.

Here, linear means there's an embedding $G \to GL_n$ for some n, and unipotent means there's an embedding $G \to \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$.

If G is reductive, then

- $\mathrm{Lie}\, G =: \mathfrak{g} = \underbrace{\mathfrak{z}_{\mathfrak{g}}}_{\text{center}} \oplus \underbrace{[\mathfrak{g},\mathfrak{g}]}_{\text{semisimple}}$
- G is semisimple $\Rightarrow$ G is reductive. ("G is semisimple" means "$\mathfrak{g}$ is semisimple.")

§1 Root data

To classify reductive groups, we need slightly more info than for semisimple Lie alg's. We'll need four pieces of data.

Def A torus is an algebraic group T over k

$$T_{\bar{k}} \cong \mathbb{G}_m^r, \text{ some } r \geq 0.$$

T is split if $T \cong \mathbb{G}_m^r$ (over k instead of $\bar{k}$)

Think: $U(1)$ over $\mathbb{R}$ is a circle, not an $\mathbb{R}^\times$, But $U(1)^{\mathbb{C}} \cong GL_1(\mathbb{C}) = \mathbb{G}_m(\mathbb{C})$. So $U(1)$ is a nonsplit torus over $\mathbb{R}$.

Fix G a connected reductive group over k, and fix a maximal torus $T \subset G$.

Assumption G is split, meaning T is split for some choice of T, which we fix.

Here are the four ingredients:

- Characters: Let $X = X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m)$. (homomorphisms of alg groups)

Remark $\mathfrak{h} := \mathrm{Lie}\, T$ is a Cartan in $\mathfrak{g} = \mathrm{Lie}\, G$, and $\mathfrak{h} \cap [\mathfrak{g},\mathfrak{g}]$ is a Cartan in $[\mathfrak{g},\mathfrak{g}]$. Then if $\chi : T \to \mathbb{G}_m$ is a character, $d\chi : \mathfrak{h} \to k = \mathrm{Lie}\, \mathbb{G}_m$ is a character of $\mathfrak{h}$.

- Roots: $G \overset{Ad}{\curvearrowright} \mathfrak{g}$, and T is simultaneously diagonalizable on $\mathrm{Lie}\, G$. Let $\Phi := \{\text{roots}\} := \{\text{simult eigenvalues for } T \overset{Ad}{\curvearrowright} \mathrm{Lie}\, G.\} \subseteq X$.

Note: Roots are trivial on $Z(G)$.

Remark If $\alpha \in \Phi$, then $d\alpha : \mathfrak{h} \to k$ is a root of $\mathfrak{g}$, i.e., $d\alpha|_{\mathfrak{h} \cap [\mathfrak{g},\mathfrak{g}]}$ is a root of $[\mathfrak{g},\mathfrak{g}]$, and $d\alpha$ is trivial on $\mathfrak{z}_{\mathfrak{g}}$.

- Cocharacters: $X^\vee = X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T)$.
- Coroots: $\forall \alpha \in \Phi$, choose root vectors $E_\alpha, E_{-\alpha} \in [\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{g}$ for $d\alpha, -d\alpha$.

  $\exists$ unipotent subgroups $U_\alpha, U_{-\alpha}$ with $\mathrm{Lie}\, U_{\pm\alpha} = k \cdot E_{\pm\alpha}$

  The subgroup H_α generated by $U_\alpha, U_{-\alpha}, T$ has derived group $[H_\alpha, H_\alpha] =: G_\alpha \cong SL_2$ or PSL_2.

  Then $\exists!\ \alpha^\vee : \mathbb{G}_m \to G_\alpha$ s.t. $\alpha \circ \alpha^\vee = (x \mapsto x^2)$. This is the coroot associated with α.

  Let $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$. This is the set of coroots.

Now define a pairing $X \times X^\vee \xrightarrow{(\cdot,\cdot)} \mathbb{Z}$ by $(\chi, \psi) = n$, where $\psi \circ \chi : \mathbb{G}_m \to \mathbb{G}_m$ is $x \mapsto x^n$. (So $(\alpha, \alpha^\vee) = 2$.)

This is a perfect pairing.

$(X, \Phi, X^\vee, \Phi^\vee)$ is an example of:

Def A root datum is a quadruple $(X,\Phi,X^\vee,\Phi^\vee)$ with $X, X^\vee$ free abelian groups, a perfect pairing $(\cdot,\cdot): X\times X^\vee\to\mathbb{Z}$, and a bijection $\Phi\to\Phi^\vee$, denoted by $\alpha\mapsto\alpha^\vee$, such that:

- $\forall\,\alpha\in\Phi$, $(\alpha,\alpha^\vee)=2$
- $\forall\,\alpha,\beta\in\Phi$,

$$s_\alpha(\beta):=\beta-(\beta,\alpha^\vee)\alpha\in\Phi, \qquad \text{and,} \qquad s_{\alpha^\vee}(\beta^\vee)=\beta^\vee-(\alpha,\beta^\vee)\alpha^\vee\in\Phi^\vee.$$

Rmk $(\alpha,\beta^\vee)$ replaces $\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}$ from the semisimple Lie algebra story. In particular, $(\alpha,\beta^\vee)\neq(\beta,\alpha^\vee)$ if α and β have different lengths.

Note: Φ need not generate X, or even $X\otimes\mathbb{R}$ over $\mathbb{R}$, due to the center.

Eg (GL_n) Let $T=\begin{pmatrix}* & & 0\\ & \ddots & \\ 0 & & *\end{pmatrix}\subseteq GL_n$. Define $e_i\in X$ by $e_i\begin{pmatrix}a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix}=a_i$. Then

$$X:=X^*(T)=\bigoplus_{i=1}^{n}\mathbb{Z}e_i \quad \text{(Writing the group law on } X \text{ additively.)}$$

Letting $\alpha_{ij}\begin{pmatrix}a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix}=a_i/a_j$ for $i\neq j$, then $\Phi=\{\alpha_{ij},\ i\neq j\}$

To find $U_{\alpha_{ij}}$ and $G_{\alpha_{ij}}$, let $E_{ij}\in \mathrm{Lie}\,GL_n=M_{n\times n}(k)$ be

$$E_{ij}=\begin{pmatrix}0 & & 1_{ij}\\ & \ddots & \\ & & 0\end{pmatrix}$$

Then

$$\mathrm{Ad}\begin{pmatrix}a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix}(E_{ij})=\begin{pmatrix}a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix}\begin{pmatrix}0 & & 1_{ij}\\ & \ddots & \\ & & 0\end{pmatrix}\begin{pmatrix}a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix}^{-1}=\begin{pmatrix}0 & & \overset{i,j\text{ position}}{a_i/a_j}\\ & \ddots & \\ & & 0\end{pmatrix}=\alpha_{ij}\begin{pmatrix}a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix}E_{ij}.$$

Note $\mathrm{Lie}\begin{pmatrix}1 & & *_{ij}\\ & \ddots & \\ & & 1\end{pmatrix}=k\cdot E_{ij}$, and so $\begin{pmatrix}1 & & *_{ij}\\ & \ddots & \\ & & 1\end{pmatrix}$ is $U_{\alpha_{ij}}$. Then

$$G_\alpha=\text{group gen by } U_{\alpha_{ij}},\ U_{\alpha_{ji}},\ \text{and } \begin{pmatrix}1 & & & & \\ & \ddots & & & \\ & & \overset{i\text{-position}}{a} & & \\ & & & \overset{j\text{-position}}{a^{-1}} & \\ & & & & \ddots\ 1\end{pmatrix}$$

is an SL_2. We find

$$\alpha_{ij}^\vee(t)=\begin{pmatrix}1 & & & & \\ & \ddots & & & \\ & & \overset{i}{t} & & \\ & & & \overset{j}{t^{-1}} & \\ & & & & \ddots\ 1\end{pmatrix},$$

so that $\alpha_{ij}(\alpha_{ij}^\vee(t))=t/t^{-1}=t^2$.

Eg (SL_2). $T=\left\{\begin{pmatrix}a & \\ & a^{-1}\end{pmatrix}\right\}\subseteq SL_2$.

Let $e_1\in X$, $e_1\begin{pmatrix}a & \\ & a^{-1}\end{pmatrix}:=a$. Then

$$X=\mathbb{Z}e_1.$$

One checks $\Phi=\{\pm\alpha_{12}\}$, but $\alpha_{12}\begin{pmatrix}a & \\ & a^{-1}\end{pmatrix}=a/a^{-1}=a^2=e_1\begin{pmatrix}a & \\ & a^{-1}\end{pmatrix}^2$, so alternatively, we can write

$$\Phi=\{\pm 2e_1\}=\{\pm\alpha_{12}\}$$

The map $a\mapsto\begin{pmatrix}a & \\ & a^{-1}\end{pmatrix}$ generates $X^\vee$. But $a\mapsto\begin{pmatrix}a & \\ & a^{-1}\end{pmatrix}\xmapsto{\alpha_{12}}a^2$, so

$$\alpha_{12}^\vee(a)=\begin{pmatrix}a & \\ & a^{-1}\end{pmatrix}.$$

Then

$$X^\vee=\mathbb{Z}\alpha_{12}^\vee$$

and

$$\Phi^\vee=\{\pm\alpha_{12}^\vee\}.$$

Note $\mathbb{Z}\Phi=2X\neq X$, but $\mathbb{Z}\Phi^\vee=X^\vee$.

Eg (PGL_2). $T = \{\begin{pmatrix} a & \\ & b \end{pmatrix}\} = \{\begin{pmatrix} c & \\ & 1 \end{pmatrix}\}$.

Note X is gen by $\begin{pmatrix} c & \\ & 1 \end{pmatrix} \mapsto c$. But now

$$\alpha_{12}\begin{pmatrix} c & \\ & 1 \end{pmatrix} = c/1 = c.$$

So

$$X = \mathbb{Z}\alpha_{12},$$

and

$$\Phi = \{\pm\alpha_{12}\}$$

$X^\vee$ is generated by $e^\vee$, where $e^\vee(c) = \begin{pmatrix} c & \\ & 1 \end{pmatrix}$.

We have $\alpha_{12}^\vee(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} = \begin{pmatrix} a^2 & \\ & 1 \end{pmatrix}$, because then $\alpha_{12}(\alpha_{12}^\vee(a)) = a^2$, so

$$\Phi^\vee = \{\pm 2e^\vee\}$$

Note now $\mathbb{Z}\Phi = X$, but $\mathbb{Z}\Phi^\vee = 2X^\vee \neq X^\vee$.

<u>Remark</u> $(X^\vee, \Phi^\vee, X, \Phi)$ w/ $(\varphi, \chi)^\vee := (\chi, \varphi)$ is again a root datum, the <u>dual root datum</u>. We see SL_2, PGL_2 are dual. GL_n is self-dual.

Let $V = \mathbb{R}\Phi \subseteq X \otimes \mathbb{R}$. Define $\langle \varphi, \varphi' \rangle = \sum_{\alpha \in \Phi} (\varphi, \alpha^\vee)(\varphi', \alpha^\vee)$. Then $(\alpha, \beta^\vee) = \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}$. Moreover $(V, \Phi, \langle \cdot, \cdot \rangle)$ is a root system, and is the root system of $[\mathfrak{g}, \mathfrak{g}]$.

<u>Thm</u> Split connected reductive groups over k are classified up to iso by their root data.

<u>§2 Parabolics</u>

<u>Def</u> A <u>Borel subgroup</u> of G is a maximal, Zariski closed, solvable, connected subgroup of G.

A <u>parabolic subgroup</u> is any subgroup containing a Borel. (Equivalently, G/Parabolic is a projective variety, Borel = minimal parabolic)

To find a Borel $B \subseteq G$, choose T, and choose "$>$" on Φ. Then T, along with U_α for $\alpha > 0$, generate B.

<u>Def</u> Let $P \subseteq G$ be a parabolic. Define $N = N_P := R_u(P) :=$ unipotent radical of P.

Given a maximal torus S in P, define also $M = M_P := Z_P(S)$, called a <u>Levi subgroup</u> of P.

<u>Facts</u>
- B contains a maximal torus in G, hence S is a maximal torus in G.
- M_P depends on S, but any two such M_P's are conjugate in P.
- $P = M \ltimes N$.
- M is reductive.
- $M_B = S$.

<u>Def</u> Given T and "$>$" on Φ, the P's, containing the Borel B constructed as above, are called <u>"standard"</u>. Then $S = T$.

Eg (GL_n) Take $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$, and choose "$>$" on Φ so that $\alpha_{i,i+1}$ are the simple roots. Then $B = \begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix}$, $N_B = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$, $M_B = T$.

Standard P's are block-upper-triangular matrices:

$$P = \begin{pmatrix} \boxed{*} & & * \\ & \ddots & \\ & & \boxed{*} \end{pmatrix}, \quad M_P = \begin{pmatrix} \boxed{*} & & 0 \\ & \ddots & \\ & & \boxed{*} \end{pmatrix}, \quad N_P = \begin{pmatrix} \boxed{I} & & * \\ & \ddots & \\ & & \boxed{I} \end{pmatrix}.$$

In general, fixing $T \subseteq G$ and "$>$" on Φ, to get all P's containing B, follow this procedure:

Take any subset $\theta \subseteq \Delta$ of simple roots. Let P_θ be generated by T and U_α for $\alpha \in \Phi$ satisfying:

If $\alpha = \sum_{\beta \in \Delta} n_\beta \beta$, then $n_\beta \geq 0$ $\forall \beta \notin \theta$. (You're allowed to negate the roots in θ.)

<u>Facts</u>:

- $M_\theta := M_{P_\theta}$ has root system with Dynkin diagram obtained from that of G by removing all nodes not in θ.
- The U_α's in M_θ are exactly those coming from α's in Φ which can be written as $\alpha = \sum_{\beta \in \theta} n_\beta \beta$.

  In particular, $U_{\pm\beta} \subseteq M_\theta$ $\forall \beta \in \theta$.
- The U_α's in $N_\theta := N_{P_\theta}$ are exactly those coming from α's in Φ which can be written as $\alpha = \sum_{\beta \in \Delta} n_\beta \beta$ with $n_\beta \geq 0$ for all $\beta \in \Delta$ and at least one $n_\beta > 0$ for some $\beta \notin \theta$. In particular, $\forall \beta \notin \theta$, we have $U_\beta \subseteq N_\theta$ and $U_{-\beta} \cap P_\theta = \{1\}$.
- $P_\Delta = G$, $P_\emptyset = B$.

<u>Eg</u> GL_4:

$T = \begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{pmatrix}$, Δ usual.

θ	$\alpha_{12} - \alpha_{23} - \alpha_{34}$	$\emptyset$	α_{12} $\quad$ α_{34}	$\alpha_{12} - \alpha_{23}$	α_{23}
P_θ	GL_4.	$\begin{pmatrix} \textcolor{green}{*} & \textcolor{red}{*} & \textcolor{red}{*} & \textcolor{red}{*} \\ & \textcolor{green}{*} & \textcolor{red}{*} & \textcolor{red}{*} \\ & & \textcolor{green}{*} & \textcolor{red}{*} \\ & & & \textcolor{green}{*} \end{pmatrix}$	$\begin{pmatrix} \textcolor{green}{*} & \textcolor{green}{*} & \textcolor{red}{*} & \textcolor{red}{*} \\ \textcolor{green}{*} & \textcolor{green}{*} & \textcolor{red}{*} & \textcolor{red}{*} \\ & & \textcolor{green}{*} & \textcolor{green}{*} \\ & & \textcolor{green}{*} & \textcolor{green}{*} \end{pmatrix}$	$\begin{pmatrix} \textcolor{green}{*} & \textcolor{green}{*} & \textcolor{green}{*} & \textcolor{red}{*} \\ \textcolor{green}{*} & \textcolor{green}{*} & \textcolor{green}{*} & \textcolor{red}{*} \\ \textcolor{green}{*} & \textcolor{green}{*} & \textcolor{green}{*} & \textcolor{red}{*} \\ & & & \textcolor{green}{*} \end{pmatrix}$	$\begin{pmatrix} \textcolor{green}{*} & \textcolor{red}{*} & \textcolor{red}{*} & \textcolor{red}{*} \\ & \textcolor{green}{*} & \textcolor{green}{*} & \textcolor{red}{*} \\ & \textcolor{green}{*} & \textcolor{green}{*} & \textcolor{red}{*} \\ & & & \textcolor{green}{*} \end{pmatrix}$

color key:
$\textcolor{green}{M_\theta}$
$\textcolor{red}{N_\theta}$

There are three other standard parabolics, obtained from $\theta = \{\alpha_{23}, \alpha_{34}\}$, $\{\alpha_{12}\}$, and $\{\alpha_{34}\}$.

<u>Lecture 4</u> Eisenstein Series. 4/23/19

§1 The definition

Let G be a split connected reductive group over $\mathbb{Q}$.

Fix $T \subseteq G$ split max'l torus, let $\Phi = \Phi(G,T) \subseteq X(T)$ the root system.

Fix $B \supseteq T$ Borel (equivalently, $\Delta \subseteq \Phi$ simple roots).

Fix $\theta \subseteq \Delta$, and let $P = P_\theta$ be the associated parabolic.

Write $P = MN$.

Let $\delta_P : M(\mathbb{A}) \to \mathbb{R}_{>0}$, $\delta_P(m) = |\det(m \overset{ad}{\curvearrowright} \operatorname{Lie} N(\mathbb{A}))|$ – modulus character. Extend to $\delta_P : P(\mathbb{A}) \to \mathbb{R}_{>0}$, trivial on N.

Let $K \subseteq G(\mathbb{A})$ a maximal compact.

<u>Def</u> For $\pi = (\pi, V_\pi) \subseteq L^2_{cusp}(M(\mathbb{Q})\backslash M(\mathbb{A}), \omega)$ a cuspidal automorphic representation of M, let

$$\tilde{I}_P^G(\pi, s) := \{ \tilde{f} : G(\mathbb{A}) \to V_\pi \mid f \text{ is smooth, } K\text{-finite, } f(pg) = \delta_P^{s+1/2}(p)\,(\pi(p) f(g)) \ \forall p \in P(\mathbb{A}),\ g \in G(\mathbb{A}) \},$$ (the normalized induction)

with action of G by right translation. Let

$$I_P^G(\pi, s) := \{ f : G(\mathbb{A}) \to \mathbb{C} \mid f(g) = \tilde{f}(g)(1) \text{ for some } \tilde{f} \in \tilde{I}_P^G(\pi, s) \}$$

(again with the same action)

<u>Rmk</u> $I_P^G(\pi, s) \cong \tilde{I}_P^G(\pi, s)$, G-equivariantly.

<u>Motivation</u> The non-cuspidal automorphic representations of should come in some way from inductions like this from all parabolics. But the functions in $I_P^G(\pi, s)$ are not automorphic forms! The Eisenstein series make automorphic forms out of these functions in an equivariant way.

<u>Def</u> Given $f \in I_P^G(\pi) := I_P^G(\pi, 0)$, and $g = pk \in G(\mathbb{A}) = P(\mathbb{A})K$, write

$$f_s(g) = \delta_P^s(p) f(g) \in I_P^G(\pi, s).$$

Then define the <u>Eisenstein series</u> by

$$E(g, f, s) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f_s(\gamma g) \quad \text{(when convergent).}$$

<u>Rmk</u> $E(g, f, s)$ is an automorphic form which transforms under $G(\mathbb{A})$ the same way as f_s.

<u>Fact</u> $E(g, f, s)$ converges for $\operatorname{Re}(s) \gg 0$ and meromorphically continues to $\mathbb{C}$.

<u>Eg</u> $G = GL_2$, $P = B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$, $\pi = 1$. Then $\delta_B\begin{pmatrix} x_1 & y \\ & x_2 \end{pmatrix} = |x_1/x_2|$.

One can choose f st., if $g = \left(1, 1, \ldots, \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix}_\infty\right)$, then

$$E(g, f, s) = \sum_{\gamma \in \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\backslash SL_2(\mathbb{Z})} \operatorname{Im}(\gamma(x+iy))^s = \sum_{(c,d)=1} \frac{y^s}{|c(x+iy)+d|^{2s}},$$

a key point being $P(\mathbb{Q})\backslash GL_2(\mathbb{Q}) \cong \mathbb{P}^1(\mathbb{Q}) \cong \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\backslash SL_2(\mathbb{Z})$.

§2. Constant terms

<u>Def</u> Let $P' = P_{\theta'} = M'N' \subseteq G$ be a standard parabolic, and ϕ an automorphic form on G. Define the constant term along P' to be

$$\phi_{P'}(g) = \int_{\underbrace{N'(\mathbb{Q})\backslash N'(\mathbb{A})}_{\text{measure } 1.}} \phi(ng)\, dn$$

<u>Def</u> The opposite parabolic to P is the parabolic $P^- = M^- N^-$ with $M^- = M$, but $N^- = \prod_{U_\alpha \subseteq N} U_{-\alpha}$.

<u>Eg</u> $\begin{pmatrix} * & \\ * & * \end{pmatrix}$ is opposite to $\begin{pmatrix} * & * \\ & * \end{pmatrix}$.

Thm (Langlands, Shahidi) Assume P, P' are maximal. Let $W(P,P') = \{w \in W(G,T) \mid w(\theta) = \theta'\}$. Then

$$E_{P'}(g,f,s) = \begin{cases} f_s(g) & W(P,P') = \{1\} \quad (\text{so } \theta = \theta') \\ \int_{N'(\mathbb{A})} f_s(wn'g)\, dn' & W(P,P') = \{w\}, \text{ but } \theta \neq \theta' \\ f_s(g) + \int_{N'(\mathbb{A})} f_s(wn'g)\, dn' & W(P,P') = \{1, w\} \\ 0 & \text{otherwise} \end{cases} \quad \text{Here, } wP'w^{-1} = P^-.$$

Idea of proof

$$E_{P'}(g,f,s) = \int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f_s(\gamma n' g)\, dn'$$

$$= \cdots = \int_{N'(\mathbb{Q}) \cap \gamma^{-1} P(\mathbb{Q}) \gamma \backslash N'(\mathbb{A})} \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})/N'(\mathbb{Q})} f_s(\gamma n' g)\, dn'$$

Use

$$G(\mathbb{Q}) = \bigcup_{w \in W(G,T)} P(\mathbb{Q}) w P'(\mathbb{Q})$$

to split this into a sum over w. Most terms vanish by cuspidality of π; what's left is what's above. :)

§3 Intertwining operators.

Rmk If $\pi = \otimes_v' \pi_v$, then $I_P^G(\pi) = \otimes_v' I_P^G(\pi_v)$.

Def For $w \in W(G,T)$, $f \in I_P^G(\pi)$, define the intertwining operators by

$$M_v(\pi_v, w)(f_v) = \int_{(N_B \cap w N^- w^{-1})(\mathbb{Q}_v)} f(w^{-1} n g)\, dn \qquad (f = \otimes_v f_v)$$

$$M(\pi, w)(f) = \int_{(N_B \cap w N^- w^{-1})(\mathbb{A})} f(w^{-1} n g)\, dn$$

Then

$$M_v(\pi_v, w): I_P^G(\pi_v) \to I_P^G({}^w\pi_v), \qquad M(\pi_v, w): I_P^G(\pi) \to I_P^G({}^w\pi), \qquad G\text{-equivariantly, where}$$

$${}^w\pi(m_v) = \pi_v(w^{-1} m_v w) \text{ and } {}^w\pi(m) := \pi(w^{-1} m w).$$

Fact If $I_P^G(\pi_v)$ is unramified and $f_v \in I_P^G(\pi_v)^{K_v}$, then $M_v(\pi_v, w) f_v \in I_P^G({}^w\pi_v)^{K_v}$

Note: If $wP'w^{-1} = P^-$, then $w^{-1} N^- w = N' = N_B \cap w^{-1} N^- w$, so

$$\int_{N'(\mathbb{A})} f_s(wn'g)\, dn' = \int_{(N_B \cap w^{-1} N^- w)(\mathbb{A})} f_s(wng)\, dn = M(\pi, w^{-1}) f_s.$$

So to compute E_P, it suffices to compute $M(\pi, w^{-1}) f_s$.

We can do this easily at unramified places.

Recall (Satake) If π_v - unr, then we get $\chi_{\pi_v}: T(\mathbb{Q}_v)/T(\mathbb{Z}_v) \to \mathbb{C}^\times$ via the Satake transform.

$I_P^G(\pi,s)$ $\ni$ normalized spherical; $f_v^0(1)=1$

Thm (Gindikin-Karpelevich, Langlands) Let $\{\infty\} \subseteq S$ = set of places away from which π and $I_P^G(\pi,s)$ are unr. Let $f_s = \bigotimes_{v \in S} f_{v,s} \otimes \bigotimes_{v \notin S} f_{v,s}^0$

Let Σ be the roots in N. View $\chi_v = \chi_{\pi_v}$ in $X(T) \otimes \mathbb{C}$. Then

$I_P^G(w^{-1}\pi_v, -s)$, normalized spherical

$$M(\pi, w^{-1}) f_s = \left(\prod_{v \notin S} \prod_{\alpha \in \Sigma} \frac{(1 - \langle \chi_v, \alpha^\vee \rangle p_v^{-s\langle 2\rho_P, \alpha^\vee \rangle})^{-1}}{(1 - \langle \chi_v, \alpha^\vee \rangle p_v^{-1-s\langle 2\rho_P, \alpha^\vee \rangle})^{-1}} \right) \left(\bigotimes_{v \in S} M_v(\pi_v, w^{-1}) f_{v,s} \right) \otimes \bigotimes_{v \notin S} f^0_{v,w^{-1},s}$$

Observation (Tits) Let $H_0 = \sum_{\alpha \in \Sigma} h_{\alpha^\vee} \in \mathrm{Lie}(T^\vee(\mathbb{C}))$, $\mathfrak{n} \subseteq \mathrm{Lie}(G^\vee(\mathbb{C}))$ spanned by $\{\alpha^\vee \mid \alpha \text{ is not in } M\}$. Let a_i, $i = 1, \dots, n$, be the eigenvalues of $\mathrm{ad}(H_0)$ on $\mathfrak{n}$, $\mathfrak{n}_i$ the eigenspaces, r_i the reps of $G^\vee$ on $\mathfrak{n}_i$. Then the product above is $\dfrac{L^S(a_i s, \pi, r_i^\vee)}{L^S(1 + a_i s, \pi, r_i^\vee)}$

§4. Examples

<u>Eg</u> $G = GL_4$, $P = P_{\{\alpha_1, \alpha_3\}} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}$. Then $M = GL_2 \times GL_2$. Let π, π' be cuspidal automorphic forms on GL_2.

Let $P' = P$. Then $W(P,P) = \{1, w\}$ where $w = s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2}$. Here,

$$w(\alpha_1) = \alpha_3, \quad w(\alpha_3) = \alpha_1, \quad w(\alpha_2) = -\alpha_1 - \alpha_2 - \alpha_3.$$

Note $w(\theta) = \theta$, $wPw^{-1} = P^- = \begin{pmatrix} * & * & & \\ * & * & & \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$, $w = w^{-1}$ (because s_{α_1} and s_{α_3} commute)

So if $f_s = \bigotimes_{v \in S} f_v \otimes \bigotimes_{v \notin S} f_v^\circ$, then

$$E_P(g, f, s) = f_s(g) + \left(\prod_i \frac{L^S(a_i s, \pi, r_i^\vee)}{L^S(1 + a_i s, \pi, r_i^\vee)} \right) \left(\bigotimes_{v \in S} M_v(\pi_v, w) f_v \right) \otimes \bigotimes_{v \notin S} f_{v,w}^\circ$$

We now compute the L-functions, i.e., we compute a_i and $r_i^\vee$.

Say, at unramified v,

$$L_v(\pi_v, s) = (1 - \beta_1 p_v^{-s})^{-1} (1 - \beta_2 p_v^{-s})^{-1}, \quad L_v(\pi_v', s) = (1 - \beta_1' p_v^{-s})^{-1} (1 - \beta_2' p_v^{-s})^{-1}.$$

Then $\chi_{\pi_v}\begin{pmatrix} p_v & \\ & 1 \end{pmatrix} = \beta_1$, $\chi_{\pi_v}\begin{pmatrix} 1 & \\ & p_v \end{pmatrix} = \beta_2$, $\chi_{\pi_v'}\begin{pmatrix} p_v & \\ & 1 \end{pmatrix} = \beta_1'$, $\chi_{\pi_v'}\begin{pmatrix} 1 & \\ & p_v \end{pmatrix} = \beta_2'$.

Let $e_i = \left(\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix} \mapsto a_i \right)$. Then, as elements of $X(T) \otimes \mathbb{C}$, $\chi_{\pi_v} = \beta_1 e_1 + \beta_2 e_2$, $\chi_{\pi_v'} = \beta_1' e_3 + \beta_2' e_4$. $\chi_P = \chi_{\pi_v} + \chi_{\pi_v'}$.

$\Sigma = \{e_1 - e_3, e_1 - e_4, e_2 - e_3, e_2 - e_4\}$, so $2\rho_P = 2(e_1 + e_2 - e_3 - e_4)$. We have

$$\langle 2\rho_P, (e_1 - e_3)^\vee \rangle = 2(1 \cdot 1 + 0 \cdot 1 + (-1)(-1) + 0 \cdot 1) = 4, \text{ and same for other elements of } \Sigma.$$

Now $\langle \chi_P, (e_1 - e_3)^\vee \rangle = \beta_1 (\beta_1')^{-1}$, $\langle \chi_P, (e_1 - e_4)^\vee \rangle = \beta_1 (\beta_2')^{-1}$, $\langle \chi_P, (e_2 - e_3)^\vee \rangle = \beta_2 (\beta_1')^{-1}$, $\langle \chi_P, (e_2 - e_4)^\vee \rangle = \beta_2 (\beta_2')^{-1}$.

Thus the L-function quotient in the constant term is a Rankin-Selberg:

$$\frac{L^S(4s, \pi \otimes (\pi')^\vee)}{L^S(1 + 4s, \pi \otimes (\pi')^\vee)} = \prod_{v \notin S} \frac{(1 - \beta_1 (\beta_1')^{-1} p_v^{-4s})^{-1} (1 - \beta_1 (\beta_2')^{-1} p_v^{-4s})^{-1} (1 - \beta_2 (\beta_1')^{-1} p_v^{-4s})^{-1} (1 - \beta_2 (\beta_2')^{-1} p_v^{-4s})^{-1}}{(1 - \beta_1 (\beta_1')^{-1} p_v^{-1-4s})^{-1} (1 - \beta_1 (\beta_2')^{-1} p_v^{-1-4s})^{-1} (1 - \beta_2 (\beta_1')^{-1} p_v^{-1-4s})^{-1} (1 - \beta_2 (\beta_2')^{-1} p_v^{-1-4s})^{-1}}.$$

So $\{i\} = \{1\}$, $a_1 = 4$, $r_1^\vee = \mathrm{Std} \otimes \mathrm{Std}^\vee$.

<u>Concluding Remarks</u>: (See Ch. 6 in Shahidi, "Eisenstein Series and Automorphic L-Functions")

- There's a formula for the constant term when P is not maximal. It's a sum over more Weyl elements. It's still in terms of intertwining operators.
- Eisenstein series can be defined for any $\nu \in X(T) \otimes \mathbb{C}$. We'd get $E(g, f, \nu)$. We just studied the case $\nu = s\rho_P$.
- $E(g, f, \nu)$ has mero. continuation and satisfies a functional equation:

$$E(g, f, s) = E(g, M(\pi, w) f, -s) \quad \text{(max'l parabolic case)}$$

or

$$E(g, f, \nu) = E(g, M(\pi, w) f, {}^w\nu) \quad \text{(general case)}.$$

This implies the meromorphic continuation and a functional equation for the constant term.

- One can try to leverage this to study analytic properties of the L-functions in the constant term. This is the Langlands–Shahidi method.
- There's a formula for M_v at $v = \infty$. You get a product of quotients of Γ-factors, in terms of a_i, r_i, and the Langlands parameters of π_∞. This is related to Harish-Chandra's c-function.